

The University of Chicago

STUDIES IN THE WARING PROBLEM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1932

By
RUTH GLIDDEN MASON

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STUDIES IN THE WARING PROBLEM

I. INTRODUCTION

In 1922, Hardy and Littlewood¹ published a proof of the existence, for every k greater than 2, of a constant $C(k)$ such that every number greater than $C(k)$ can be expressed as the sum of $(k-2)2^{k-1} + 5$ k th powers. The value of $C(k)$ was not determined. In this paper we shall determine a value for $C(5)$. The result will show that every number greater than 10^{153} is the sum of fifty three fifth powers.

In determining $C(5)$ we follow in general the exposition of the Hardy-Littlewood proof given in Landau's "Vorlesungen Über Zahlentheorie."² In order to make $C(5)$ as small as possible the existence proof was somewhat modified. There is some change made in the order of presentation and in some cases, theorems were replaced by similar ones which, however, do not require as large constants. In the existence proof a single ϵ was used throughout; in order to get a lower value of $C(5)$ than would be possible if we used but one ϵ we use as many as forty-five ϵ 's. The divisions of subject matter follow the divisions in the book.

¹R.H. Hardy and E.J. Littlewood, Some Problems of Partitio Numerorum IV: The singular series in Waring's Problem and the value of the number $G(k)$, Mathematische Zeitschrift, XII (1922), pp. 161-188.

²Edmund Landau, Vorlesungen Über Zahlentheorie, Hirzel, Leipzig, (1927), pp. 235-303.

II. PRELIMINARY THEOREMS

All references will be understood to refer to the previously mentioned book, Landau's "Vorlesungen Über Zahlen-theorie."

Our value for k is 5, and consequently, the notations defined about two thirds of the way down page 237 give us

$$a = 1/5, \quad K = 16, \quad \chi = 15/16.$$

We do not use α as defined on page 247, but use instead

$$\alpha/16 = .0124999.$$

For s we take its lowest value, which is 53.

The first constant to be evaluated is c in Theorem 258 on page 248.

Theorem 258: On every arc B, for $-\pi \leq \theta \leq \pi$,

$$(276) \quad |1-X| \geq e^{-1} \sqrt{(1/n^2) + \theta^2}.$$

In the third line of proof on page 249 we have

$$\sqrt{(1-e^{-1/n})^2 + 4e^{-1/2} \sin^2(\theta/2)} \geq c \sqrt{(1/n^2) + \theta^2}.$$

Therefore

$$n(1-e^{-1/n}) > c; \quad e^{-1/4} \sin(\theta/2)/(\theta/2) > c.$$

We find by the usual differentiation process that

$$\text{Minimum } n(1-e^{-1/n}) = e^{-1/n} > e^{-1}.$$

$$\text{Minimum } e^{-1/4} \sin(\theta/2)/(\theta/2) = \pi/2 e^{-1/4}.$$

$$e^{-1} < (\pi/2)e^{-1/4} ;$$

$$\therefore c = e^{-1} .$$

Theorem 259 is accepted as it stands on page 250 .

Theorem 260. If $T(m)$ is the number of divisors of $m > 0$, then for every $\varepsilon_1 > 0$

$$T(m) < C_6 m^{\varepsilon_1} ,$$

where C_6 depends only on ε_1 , and

$$C_6 \leq \prod_{p < 2^{1/\varepsilon_1}} p^{\varepsilon_1 / (e \varepsilon_1 \log p)} .$$

Proof:

Let $\varepsilon_1 < 1$, $m > 1$, and the canonical representation be

$$m = \prod_{p|m} p^{\ell} .$$

Then

$$T(m) = \prod_{p|m} (\ell + 1) .$$

$$[T(m)]/m^{\varepsilon_1} = \prod_{p|m} (\ell + 1)/p^{\ell \varepsilon_1} \leq \prod_{\substack{p|m \\ p < 2}} (\ell + 1)/p^{\ell \varepsilon_1}$$

$$\leq \prod_{\substack{p|m \\ p < 2^{1/\varepsilon_1}}} \text{Max} [(x+1)/p^{x \varepsilon_1}] ,$$

where x is a continuous variable > 0 . By the usual differentiation method we can show that $(x+1)/p^{x \varepsilon_1}$ has a maximum value when

$$x = (1/(\varepsilon_1 \log p)) - 1 .$$

$$\text{Max } (x+1)/p^{x \varepsilon_1} = p^{\varepsilon_1 / (e \varepsilon_1 \log p)} .$$

$$T(m)/m^{\varepsilon_1} \leq \prod_{\substack{p|m \\ p < 2}} p^{\varepsilon_1 / (e \varepsilon_1 \log p)} \leq \prod_{p < 2^{1/\varepsilon_1}} p^{\varepsilon_1 / (e \varepsilon_1 \log p)} .$$

Let the number of ways G can be factored into g factors be represented by $T_g(G)$.

(Our next theorem is similar to but different from theorem 261 in the book.)

Theorem 261. The number of solutions $T_4(m_1)$ of the diophantine equation

$$h_1 h_2 h_3 h_4 = m_1 ,$$

with

$$0 < h_i < m, (i=1..4), m_1 > 0 ,$$

is

$$< C_6(\varepsilon_3) C_6(\varepsilon_4) C_6(\varepsilon_5) m^{\varepsilon_2} .$$

Proof:

$$T_2(m^4) < C_6(\varepsilon_3) m^{4\varepsilon_3} .$$

$$\therefore T_2(m_1) < C_6(\varepsilon_3) m_1^{\varepsilon_3} \leq C_6(\varepsilon_3) m^{4\varepsilon_3} .$$

$\therefore C_6(\varepsilon_3) m^{4\varepsilon_3} > \text{number of ways } m_1 \text{ can be factored into two factors } m_2, m_3 \text{ such that } m_2 < m^2, m_3 < m^2 .$

$$T_2(m^2) < C_6(\varepsilon_4) m^{2\varepsilon_4} ;$$

$$T_2(m_2) < C_6(\varepsilon_4) m_2^{\varepsilon_4} \leq C_6(\varepsilon_4) m^{2\varepsilon_4} ;$$

$$T_2(m_3) < C_6(\varepsilon_5) m_3^{\varepsilon_5} \leq C_6(\varepsilon_5) m^{2\varepsilon_5} ;$$

$$T_4(m_1) < C_6(\varepsilon_3) \cdot C_6(\varepsilon_4) \cdot C_6(\varepsilon_5) m^{\varepsilon_2} ,$$

where

$$\varepsilon_2 = 4\varepsilon_3 + 2\varepsilon_4 + 2\varepsilon_5 .$$

Theorem 262:

1) The number of solutions $R(m)=R(m,k)$ of the diophantine equation

$$h_1^5 + h_2^5 = m, h_1 \geq h_2 \geq 0,$$

for $m > 0$, satisfies

$$R(m) < 4C_6(\varepsilon_6) m^{\varepsilon_6}.$$

2) Further for $m > 0$

$$\sum_{\nu=0}^m R(\nu) \leq 4m^{2/5}.$$

3) Finally for $m > 0$

$$W(m) = \sum_{\nu=0}^m R^2(\nu) < 16C_6(\varepsilon_7) m^{2a+\varepsilon_7}.$$

In the proof we consider case (12) since our $k=5$, an odd number.

From the twelfth line of page 252 we have

$$R(m) \leq 4C_6(\varepsilon_6) m^{\varepsilon_6}.$$

From the line just above theorem 263 on page 252 we have

$$\sum_{\nu=0}^m R^2(\nu) \leq \max_{0 \leq \nu \leq m} R(\nu) \sum_{\nu=0}^m R(\nu) < 16C_6(\varepsilon_7) m^{(2/5)+\varepsilon_7}.$$

Theorem 263 is used exactly as it appears on page 252.

Theorem 264. For $m > 0$, r integral, λ real,

$$S = \sum_{h=r+1}^{r+m} e^{2\pi i \lambda h^5},$$

Then

$$(285) \quad S^{16} < 3^4 2^{11} (m^{15} + m^{11} \sum_{h_1 \cdots h_4=1}^m \text{Min}(m, 1/\{120 h_1 \cdots h_4\})) .$$

The only difference, other than the replacement of K and k , between our (285) and that in the book is in the factor before the parenthesis. Let us call that factor t_5 and such a factor t_k for the case of k .

$$\begin{aligned}
S 2^{k-1} &= t_{k-1} (2m)^{2^{k-2}-1} \sum_{h=-(m-1)}^{m-1} m^{2^{k-2}} + m^{2^{k-2}-i+1} \sum_{h_2 \dots k_k=1}^m \text{Min} \\
&= t_{k-1} (2m)^{2^{k-2}-1} (2m \cdot m^{2^{k-2}-1} + m^{2^{k-2}-k+1} \sum \text{Min}) \\
&< t_{k-1} 2^{2^{k-2}-1} (3m^{2^{k-2}} + m^{2^{k-2}-k+1} \sum \text{Min}) \\
&< 3 \cdot 2^{2^{k-2}-1} t_{k-1} (m^{2^{k-1}} + m^{2^{k-1}-k} \sum \text{Min}) .
\end{aligned}$$

The above relation gives the recursion formula

$$\begin{aligned}
t_k &= 3 t_{k-1} \cdot 2^{2^{k-2}-1} . \\
t_1 &= 1, \quad t_2 = 3, \quad t_3 = 3^2 \cdot 2 . \\
t_4 &= 3^3 \cdot 2^4, \quad t_5 = 3^4 \cdot 2^{11} .
\end{aligned}$$

Theorem 266. If $m > 0$, r integral, λ real, then
 $|S|^{16} < 3^4 \cdot 2^{11} C_6(\xi_8) C_6(\xi_9) C_6(\xi_{10}) m^{\xi_{11}} (m^{15} + m^{11} \sum_{r=1}^{120m^4} \text{Min}(m, 1/2\{\lambda^r\}))$.

To prove this theorem we merely have to apply our theorem 261.

Theorem 267. If $m > 0$, r integral, ρ a primitive qth root

$$\begin{aligned}
S &= \sum_{h=r+1}^{r+m} \rho^{h^5}, \\
S^{16} &< C_{18} m^{\xi_{11}} \xi_{12} (m^{15} + (m/q)^{11} q),
\end{aligned}$$

where

$$C_{18} = 480 C_{19}(\xi_{12}) C_6(\xi_8) C_6(\xi_9) C_6(\xi_{10}) \cdot 3^4 \cdot 2^{11} .$$

Proof:

According to the seventh row of page 256, each subsum has

$$\leq 120 (m/q+1)^4$$

terms.

Four lines below is shown

$$\sum_{j=0}^{q-1} \text{Min}(m, 1/\{j/q\}) \leq m+2q(1+\log q) \leq C_{19} q^{1+\xi_{12}},$$

where

$$C_{19} = 2/\xi_{12}^{1-\xi_{12}}.$$

Putting all together we see that

$$\begin{aligned} S^{16} &< 480 C_{19}(\xi_{12}) C_6(\xi_8) C_6(\xi_9) C_6(\xi_{10}) m^{\xi_{10}} q^{\xi_{10}} (m^{15} + (m^{16}/q) + m^{11} q) \\ &= C_{18} m^{\xi_{11}} q^{\xi_{12}} (m^{15} + (m^{16}/q) + m^{11} q). \end{aligned}$$

Theorem 268. If ρ is a primitive q th root, $0 < m < q$,

$$S = \sum_{h=r+1}^{r+m} \rho^{h^k} < C_2' q^{15/16 + \xi_{13}}.$$

This could be proved directly as a corollary of theorem 267 with

$$S < C_{18} q^{1/16} q^{(15/16)(\xi_{11} + \xi_{12})/16}.$$

But we choose instead to follow a more complicated method which yields a lower constant multiplier, namely

$$C_2' = [3^4 \cdot 2^{11} (1 + C_6(\xi_8) C_6(\xi_9) C_6(\xi_{10}) + C_{19}(\xi_{13}))]^{1/16}.$$

First we prove this Lemma:

The congruence, $v \equiv 0 \pmod{q}$ has $\leq 5: T_4(q) m^4/q$ solutions

where

$$v = 120 h_1 \cdots h_4$$

as in Theorem 266.

Notation:

$$(120, q) = D.$$

$$H = h_1 h_2 h_3 h_4.$$

$T_\alpha(\beta)$ indicates the number of ways β can be expressed as the product of α factors.

If

$$H \equiv 0 \pmod{q/D}, \quad v \equiv \pmod{q}.$$

If $d_i < m$, and is a divisor of q/D , there are $[m/d_i]$ numbers $< m$ which are multiples of d_i . If $d_1 \cdots d_4$ is one of the $T_4(q/D)$ factorizations of q/D , then there are

$$[m/d_1] [m/d_2] [m/d_3] [m/d_4] \leq m^4/(q/D)$$

ways of forming a number,

$$M_1 d_1 M_2 d_2 M_3 d_3 M_4 d_4 \equiv 0 \pmod{q/D},$$

where

$$M_i d_i < m.$$

$\therefore$ In all there are not more than

$$m^4/(q/D) T_4(q/D) \leq 120 m^4/q T_4(q)$$

values of v which are $\equiv 0 \pmod{q}$.

Proof of Theorem:

Inserting $\lambda = \ell/q$, $(\ell, q) = 1$ in inequality (285) we get

$$S^{16} < 3^4 \cdot 2^{11} (m^{15} + m^{11} \sum_{h_1 \cdots h_4}^m \text{Min}(m, 1/\{120 \ell h_1 \cdots h_4/q\})).$$

Case I.

$$\text{Min}(m, 1/\{120 \ell h_1 \cdots h_4/q\})$$

is always m when $v \equiv 0 \pmod{q}$. By our preceding lemma and theorem 261' this can occur at most

$$120 C_6(\varepsilon_{14}) C_6(\varepsilon_{15}) C_6(\varepsilon_{16}) m_1^{4+\varepsilon_{17}q-1}$$

times, where

$$\varepsilon_{11} = 4\varepsilon_8 + 2\varepsilon_9 + 2\varepsilon_{10}.$$

Case II.

Now consider the case $H \not\equiv 0 \pmod{q/D}$.

$$\text{Let } H = H_1 h_4.$$

For a particular H_1 , let $(120H_1, q) = d$.

If h_4 is divisible by q/d , $\forall \equiv 0 \pmod{q}$, and we would have Case

I. As h_4 ranges over all other values from 1 to m , $120H/d$ ranges over not more than d sets of values $1 \dots q/d-1 \pmod{q/d}$.

$$\begin{aligned} \sum_{h_4=1}^{q/d-1} 1/\{120 h_4(H_1/d)/(q/d)\} &= 2 \sum_{1 \leq j \leq q/(2d)} 1/\{j/(q/d)\} \\ &= 2 \sum_{1 \leq j \leq q/(2d)} 1/(j/(q/d)) \leq 2 \sum_{j=1}^{q/d} (q/d)/j \leq 2q/d(1+\log(q/d)) \\ &< (2/(\varepsilon_{18}^{1-\varepsilon_{18}}))(q^{1+\varepsilon_{18}/d}) . \end{aligned}$$

Call

$$2/(\varepsilon_{18}^{1-\varepsilon_{18}}) = C_{19}(\varepsilon_{18}) .$$

There are not more than m^3 different values for H_1 (counting repeated values as separate values).

$$S^{16} < 3^4 2^{11} [m^{15} + m^{11} (C_6(\varepsilon_{14}) C_6(\varepsilon_{15}) C_6(\varepsilon_{16}) m^{5+\varepsilon_{17}/q} + C_{19} m^3 q^{1+\varepsilon_{18}})]$$

$$< 3^4 \cdot 2^{11} [1 + C_6(\varepsilon_8) C_6(\varepsilon_9) C_6(\varepsilon_{10}) + C_{19}(\varepsilon_6) q^{15+16\varepsilon_{13}}] .$$

We could derive theorem 269 at once, but we can get a much better value if we turn to page 298 of the book.

Theorem 314. For

$$\left| T_{\ell} \right| \leq \begin{cases} q = p & \ell \geq 1 \\ 1 & \text{for } p > 102, \\ 4/p^{3/10} & \text{for } p < 102, \ell \equiv 1 \pmod{5} , \\ 1 & \text{for } p \neq 5, \ell = 2, 3, 4, 5 \pmod{5}, \\ 5 & \text{for } p = 5. \end{cases}$$

We use part (1) of the proof as it stands in the book.

Part (2) in the book gives, if

$$\ell \equiv 1 \pmod{5}, p \nmid 5, p > 102, \\ |T_p| \leq 4/p^{3/10} \leq 1.$$

Part (3) gives for the case of 5

$$|T_p| \leq 5.$$

Theorem 315:

$$|T_p| < 2^{19}. \\ |S_p| < 2^{19} q^{4/5}.$$

The top line of page 299 enables us to evaluate

$$|T_p| = \prod_{v=1}^m |T_{p_v}| \leq \prod_{p/q} |T_p| \leq 4^{25.5} \prod_{p < 102} p^{3/10} \leq 2^{19}.$$

We now have a revised

Theorem 269:

$$|S_p| < 2^{19} q^{4/5}.$$

Theorem 270:

$$A(q) < 2^{1007} q^{-48/5}.$$

Proof:

$$A(q) \leq \sum_{p(q)} |S_{p/q}|^{53} = q (2^{19} q^{4/5}/q)^{53} = 2^{1007} q^{-48/5}.$$

Theorem 273:

Let $\varepsilon_{19} > 0$, $1 \leq m \leq n^{1/5 + \varepsilon_{19}}$, ρ a primitive q th root of unity, $q \leq n^{1/5}$, θ real,

$$1/n (n^{1/5}/q)^{16/17} \leq |\theta| \leq \pi/120qm^4,$$

$$S = \sum_{h=1}^m (\rho e^{i\theta h^5}).$$

Then

$$S < c_{25} n^{(16/85) + \varepsilon_{20}},$$

where

$$C_{25} = [3^4 \cdot 2^{11} (1 + 4C_{19}(\varepsilon_{21}) + C_{29}(\varepsilon_{22}, \varepsilon_{25}))]^{1/16},$$

$$\varepsilon_{20} = [\text{Max}(15\varepsilon_{19} - (1/85), (\varepsilon_{21}/5) + 15\varepsilon_{19} - (1/85),$$

$$(1/5 + \varepsilon_{19})(\varepsilon_{25} + \varepsilon_{22}) + 11\varepsilon_{19}]^{1/16}.$$

In this paper there is some modification of the proof as given in the book. We omit the first three lines and then continue toward the bottom of page 259 for six lines.

Taking the case $j \neq 0$, let

$$H_1 = h_1 h_2 h_3, \quad H = H_1 h_4$$

and let

$$(120H_1, q) = d_1.$$

If h_4 is divisible by q/d , $v \equiv 0 \pmod{q}$ and we would have an excluded case. As h_4 ranges over all other values from 1 to m $120 H/d$ ranges over not more than $1 + [m/q]d$ sets of values congruent to $1 \dots (q/d) - 1 \pmod{q/d}$.

$$0 < j/(2q/d) = (j/2)/(q/d) \leq ((2j-1)/2)/(q/d)$$

$$\leq (j/(q/d)) + (\theta v/2) \leq (2j+1)/(2q/d)$$

$$\leq (j+(q/d))/(2q/d) = 1/2 + j/(2q/d) < 1,$$

$$(q/d) \left(\sum_{1 \leq j \leq q/(2d)} \frac{1}{j} + \sum_{q/(2d) \leq j \leq (q/d)-1} \frac{1}{j} \right)^{1/(q/d)-j}$$

$$= 2q/d \sum_{1 \leq j \leq q/2} \frac{1}{j} < 2C_{19}(\varepsilon_{21}) q^{1+\varepsilon_{21}/d},$$

we use, as first found on page 253,

$$(285) \quad S_{16} < 3^4 \cdot 2^{11} (m^{15} + m^{11} \sum_{h \cdot h_4 = 1}^m \text{Min}(m, 1/\{\lambda k! h_1 \dots h_4\})),$$

and the total contribution to it of terms for which $j \neq 0$, is

Case a)

$$m > q \\ \leq 2(md/q)m^3 (2q/d)C_{19}(\varepsilon_{21})_q^{\varepsilon_{21}} = 4C_{19}(\varepsilon_{21})m^3 q^{\varepsilon_{21}} ;$$

Case b)

$$m \leq q \\ \leq m^3 d(2q/d)C_{19}(\varepsilon_{21})_q^{\varepsilon_{21}} \leq 2C_{19}(\varepsilon_{21})m^3 q^{1+\varepsilon_{21}} ;$$

in either case the total contribution of terms for which $j \neq 0$ is

$$\leq 4C_{19}(\varepsilon_{21})n^{4/5+\varepsilon_{21}/5+4\varepsilon_{19}} .$$

The greatest number of times any number, including any number $\equiv 0 \pmod{q}$, can occur is according to theorem 261',

$$\leq C_6(\varepsilon_{22})C_6(\varepsilon_{23})C_6(\varepsilon_{24})m^{\varepsilon_{25}} = C_8(\varepsilon_{25})m^{\varepsilon_{25}} ,$$

where

$$\varepsilon_{25} = 4\varepsilon_{22} + 2\varepsilon_{23} + 2\varepsilon_{24} .$$

To evaluate the total contribution of terms for which $j=0$ we follow the method in relation 298, page 260. The total contribution is

$$\leq C_8(\varepsilon_{25})m^{\varepsilon_{25}} (2\pi/|\theta|) \sum_{r=1}^{[120m^4]} 1/q^r \\ \leq C_8(\varepsilon_{25})m^{\varepsilon_{25}} (2\pi/|\theta|) C_{19}(\varepsilon_{22}) [120m^4/q]^{\varepsilon_{22}} \\ \leq 2 C_8(\varepsilon_{25})C_{19}(\varepsilon_{22}) 120^{\varepsilon_{22}} n^{(q/n^{1/5})^{16/17} (1/q^{1+\varepsilon_{22}})^m \varepsilon_{25} + 4\varepsilon_{22}} \\ \leq C_{29}(\varepsilon_{22}, \varepsilon_{25}) n^{69/85} n^{(1/5+\varepsilon_{19})(\varepsilon_{25}+4\varepsilon_{22})} .$$

Putting all together we get

$$\leq 16 \cdot 3^4 \cdot 2^{11} (n^{3+15\varepsilon_{19}+(1/5)+11\varepsilon_{19}} (4C_{19}(\varepsilon_{21})n^{(4/5)+(\varepsilon_{21}/5+4\varepsilon_{19})} \\ + C_{29}(\varepsilon_{22}, \varepsilon_{25})n^{69/85+(1/5+\varepsilon_{19})(\varepsilon_{25}+4\varepsilon_{22})} \\ < 3^4 \cdot 2^{11} (1+4C_{19}(\varepsilon_{21})+C_{29}(\varepsilon_{22}, \varepsilon_{25}))n^{256/85+\varepsilon_{26}} ,$$

where

$$\varepsilon_{26} = \text{Max}(15\varepsilon_{19}-1/85, \varepsilon_{21}/5+15\varepsilon_{19}-1/85 ,$$

$$(1/5 + \varepsilon_{19})(\varepsilon_{25} + 4\varepsilon_{22}) + 11\varepsilon_{19} \cdot$$

$$S < C_{25} n^{16/85 + \varepsilon_{20}},$$

where

$$C_{25} = [3^4 \cdot 2^{11} (1 + 4C_{19}(\varepsilon_{21}) + C_{29}(\varepsilon_{22}, \varepsilon_{25}))]^{1/16}$$

$$\varepsilon_{20} = \varepsilon_{26}/16 \cdot$$

III. MAJOR AND MINOR ARCS

Theorem 274: On the entire circle $x = e^{-1/n}$,

$$\left| f(x) - \sum_{h=0}^{\omega} x^{h^5} \right| < 1.05 (2/(25 \varepsilon_{27} e))^{1/25} \varepsilon_{27} + 1,$$

where

$$\omega = [n^{1/5 + \varepsilon_{27}}] .$$

In the last line of the proof, constants appear,

$$n^{1/5} e^{-(1/2)n^5 \varepsilon_{27}} \int_0^{\infty} e^{-z^5/2} dz$$

$$\leq (2^{1/5}/5) \cdot (\text{Max } n^{1/5} e^{-n^5 \varepsilon_{27}/2}) \Gamma(1/5) .$$

$$\therefore \sum_{h=\omega+2}^{\infty} e^{-h^5/n} = 1.05 (2/(25 \varepsilon_{27} e))^{1/25} .$$

Theorem 275: Let ρ be any chosen root of unity, than for an integral $j \geq 0$,

$$(300) \quad \tau(j) = \sum_{h=0}^{[j^{1/5}]} \rho^{h^5} ,$$

and let $\rho^{-1}x = X$, then on the entire circle $x = e^{-1/n}$

$$(301) \quad f(x) = (1-X) \sum_{j=0}^{\infty} (j) X^j ,$$

and

$$f(x) - (1-X) \sum_{j=0}^{\omega^5} \tau(j) X^j - \tau(\omega^5) X^{\omega^5+1} < 1 + 1.05 (2/(25 \varepsilon_{27} e))^{1/(25 \varepsilon_{27})} ,$$

where

$$\omega = [n^{1/5 + \varepsilon_{27}}] .$$

Theorem 276. On the minor arcs

$$f(x) < (11+3C_{1/3}^{1/16}) n^{(1/5)-(\alpha/16)+\varepsilon_m}$$

$$n > 10^{16000}, \quad \varepsilon_{27} \geq 1/25000.$$

In the seventh line from the bottom of page 262 we have in accordance with theorem 266, as stated in this paper

$$\begin{aligned} |\tau(j)|^{16} &< C_{18}' 2^{\varepsilon_{11} n^{(1/5)+\varepsilon_{27}}} \varepsilon_{12}^{\varepsilon_{11} n^{(1/5)+\varepsilon_{27}}} (2n^{(1/5)+\varepsilon_{27}})^{15} \\ &\quad + (2n^{(1/5)+\varepsilon_{27}})^{16/q} + (2n^{(1/5)+\varepsilon_{27}})^{11/q} \\ &< C_{18}' 2^{\varepsilon_{11} n^{(1/5)+\varepsilon_{27}}} \varepsilon_{11} (2^{15} n^{3+15\varepsilon_{27}+(1-\alpha)\varepsilon_{12}} \\ &\quad + 2^{16} n^{(16/15)+16\varepsilon_{27}(1-\varepsilon_{12})\alpha} + 2^{11} n^{(11/5)+11\varepsilon_2+(1+\varepsilon_{12})(1-\alpha)}) \\ &< C_{18}' 2^{16+\varepsilon_{11} n^{(16/5)+\varepsilon_{28}-\alpha}}, \\ \varepsilon_{28} &= ((1/5)+\varepsilon_{27})\varepsilon_{11} + \max(15\varepsilon_{27}+(1-\alpha)\varepsilon_{12}, 16\varepsilon_{27}+\alpha\varepsilon_{12}). \\ |\tau(j)| &< 3C_{18}'^{1/16} n^{(1/5)-\alpha+(\varepsilon_{28}/16)}, \\ f(x) &< 1+1.05(2/(25\varepsilon_{27}e))^{1/(25\varepsilon_{27})} 3C_{18}'^{1/16} |1-x| n^{(1/5)-\alpha+\varepsilon_{28}/16} \\ &\quad + 3C_{18}'^{1/16} n^{(1/5)-\alpha+(\varepsilon_{28}/16)} < 11+3C_{18}'^{1/16} n^{(1/5)-\alpha+\varepsilon_m}. \end{aligned}$$

We are going to use Theorem 277 for two particular values of β and we will work out each case separately.

Theorem 277: With $j > 0$

a) $\left| (\Gamma(6/5)+j)/j! \right| - j^{1/5} < (3/5)\Gamma(11/5)j^{-4/5};$

b) $\left| (\Gamma(53/5)+j)/j! \right| - j^{48/5} < 53\Gamma(58/5)j^{48/5}.$

a) Call

$$\Gamma((6/5)+j)/(j!j^{1/5}) = \Phi(j).$$

As in 303 on page 264

$$\lim_{j \rightarrow \infty} \phi_1(j) = 1.$$

$$\phi_1(j+1)/\phi_1(j) = ((6/5)+j)/(j+1)(j/(j+1))^{1/5}.$$

This ratio would vary continuously if j varied

over all positive values. If for no value of $j > 0$ it has the value unity, the fraction is either always greater than zero or less than zero. If the function were equal to zero

$$(6/5+j)^5 j = (j+1)^6.$$

Expanding we get

$$(6/5)^5 j + (10 + (46/125))j^2 + (17 + (7/25))j^3 + (44 + (2/5))j^4 + 6j^5 + j^6 \\ = 1 + 6j + 15j^2 + 20j^3 + 15j^4 + 6j^5 + j^6,$$

which obviously has no real roots.

As j increases $\phi(j)$ either monotonically increases or monotonically decreases.

$$\phi(1) = \Gamma(11/5) > 1.$$

We now need the expansion of

$$(1 + (1/5v)) (1 - (1/v))^{1/5} = 1 - (3/25v^2) + A_1,$$

where A_1 is a series with all negative coefficients, and involving no powers of v greater than v^{-3} .

$$v^2 [1 - (1 + (1/5v)) (1 - (1/v))^{1/5}] = (3/25) + A_1 v^2 < 3/25$$

for all values of v . As in the sixth line of page 265

$$\left| ((6/5)+j)/(j:j^{1/5}) - 1 \right| = \Gamma(11/5) - (3/25)(1/j).$$

b) Second case is argued exactly as the first.

Call

$$\Gamma((53/5)+j)/(j:j^{53/5}) = \phi_2(j),$$

$$\lim_{j \rightarrow \infty} \phi_2(j) = 1 ,$$

$$((53/5)+j)^5 j^{48} = (j+1)^{53} ,$$

when expanded is

$$j^{53} + 53j^{52} + 10(53/5)j^{51} + 10(53/5)^2 j^{50} + 5(53/5)^3 j^{49} + (53/5)^4 j^{48} \\ = j^{53} + 53j^{52} + 1378 j^{51} + 23426j^{50} + 292825j^{49} + 2869685j^{48} + P_1 ,$$

where P_1 is a polynomial with all positive coefficients. Obviously there can be no positive real roots.

$$\phi_2(1) = (58/5) > 1 .$$

$$(1+(48/5v))(1-(1/v))^{48/5} = (1+(48/5v))(1-(48/5v)(2064/(50v^2)) - (78432/(750v^3)) \\ + A_2) ,$$

where A_2 is a series in inverse powers of v of degree four or more, in which every term after the sixth has positive coefficients, and in which the sum of first six terms is positive for $v \geq 2$. In the following relation A_3 is always positive

$$v^2[1-(1+(48/5v))(1-(1/v))^{48/5} < (1272/25) + (2703/2500) - A_3 < 53 .$$

$$\left| \Gamma((53/5)+j)/(j!j^{48/5}) - 1 \right| < 51\Gamma(58/5)j^{-1} .$$

Theorem 278: On M

$$\left| f(x) - \psi_c(x) \right| < C_5 n^{(16/85) + \epsilon_M} .$$

We follow the method of proof in the book. On page 266, we evaluate by theorem 277 the inequality below (308)

$$\text{for } j > 0, \quad ((\Gamma(6/5)+j)/j!)j^{1/5} < (3/5)\Gamma(11/5) < 1 ,$$

$$\text{for } j = 0, \quad < \Gamma(6/5) < 1 ,$$

$$\text{for } j \geq 0, \quad < 1 ,$$

which gives us

$$(310) \quad \left| \gamma(j) - j^{1/5} S_{e/q} \right| < 1 ,$$

which in turn gives

$$(311) \quad |\tau(j) - \nu(j)| < (2C_2(\varepsilon_{34})+1)q^{(15/16)+\varepsilon_{34}},$$

and which enables us to get the result six lines from the bottom of page 266 ,

$$|f(x) - \psi(x)| < 5C_2(\varepsilon_{24})n^{(16/85)+\varepsilon_{34}} \alpha - (6/8245) ,$$

In part (2) of the proof we have the value of ε_{19} as determined by the expression four lines below (314), near the middle of page 267.

$$\pi/(120n^{4/5}n^{\varepsilon_{19}}) > 2\pi/n^{389/485} ,$$

if

$$(1/240)n^{(1/485)} - 4\varepsilon_{19} \geq 1 ,$$

which is possible only if

$$\varepsilon_{19} \leq 1/1944 , \quad n \geq 240^{235,710} = C_{24} .$$

If

$$n \leq C_{24}$$

$$|f(x) - \psi(x)| \leq C_{24}^{1/5} \Gamma(1/5) + \Gamma(6/5)1/(1-e^{-1/C_{24}})^{1/5} ,$$

which is very large but will not have to be used.

Three lines from the bottom of page 267 we have

$$(315) \quad \tau(j) < [1+C_{25}(\varepsilon_{20})]n^{(16/85)+\varepsilon_{29}}$$

The last line of page 267 is evaluated to read

$$f(x) - \sum_{h=0}^{\omega} x^{h^5} < 1.05(2/(25\varepsilon_{27}e))^{1/(25\varepsilon_{27})} + 1 .$$

Since

$$(1-R) \sum_{j=0}^{\infty} R^j = 1 ,$$

$$f(x) < ([1.05(2/(25\varepsilon_{27}e))^{1/(25\varepsilon_{27})}+1]240^{-44368}+2[1+C_{25}(\varepsilon_{20})])n^{(16/85)+\varepsilon_{29}}$$

According to the seventh and eighth lines from the bottom of page 268,

$$\begin{aligned} \Psi_e(x) &< \Gamma(6/5)C_2(\xi_{35})e^{1/5}q^{(171/1360)+\xi_{35}}n^{69/425} \\ &< \Gamma(6/5)C_2(\xi_{35})e^{1/5}n^{(16/85)+\alpha\xi_{35}-(41/7860)} . \end{aligned}$$

$$\therefore |f(x) - \Psi_e(x)| < C_5 n^{(16/85)+\varepsilon_M},$$

where

$$\begin{aligned} C_5 &= \text{Max} (5C_2(\xi_{24}), (1.05(2/(25\xi_{27}e))^{1/(25\xi_{27})}240^{-44368} \\ &\quad + 2[1+C_{25}(\xi_{20})]+\Gamma(6/5)C_2(\xi_{35})e^{1/5})) , \\ &= \text{Max} (\alpha\xi_{34}-(6/8245), \xi_{20}, \alpha\xi_{35}-(41/7860)) . \end{aligned}$$

IV. THE FIRST HARDY-LITTLEWOOD THEOREM

Theorem 274: For $s > 52$,

$$\left| r(n) - ((\Gamma(6/5))^s / \Gamma(s/5)) n^{(s/5)-1} \right| < b_7 n^{(s/5)-1}.$$

For the case $s = 53$,

$$\left| r(n) - ((\Gamma(6/5))^{53} / \Gamma(53/5)) n^{(48/5)} \right| < (1/2) 10^{10^{150}} n^{(48/5) - (1/200)}.$$

In the proof of the first half of the first step we do not use the method that results in relation (320) on page 269, but instead obtain the result (321) about the middle of page 270. In the relation above (322) about the middle of page 270 we have according to theorem 276, for $s = 53$

$$\max |f^{49}(x)| \leq (11 + 3C_{18}^{1/16} (\epsilon_m))^{49} n^{(49/5) - (49 \times 16) - 1 - 49\epsilon_m}.$$

In order to evaluate C_{46} in the first expression on page 271 we must turn back to theorem 224 on page 192 of Landau's

book, which in our notation would be written

$$\sum_{m=1}^{\infty} m^{(2/5) + \epsilon_{44}} (e^{-2/n})^m < A_4 / (1 - e^{-2/n})^{1 + (2/5) + \epsilon_{44}},$$

where

$$A_4 < e^{(4/5) + 2\epsilon_{44}}.$$

In the proof, the first constant to be evaluated is A_5 obtained

in the series of relations in the middle of page 192 by setting

$$\varepsilon_{44} + (2/5) = \varepsilon_{45} ,$$

$$\varepsilon_{45}^2 \sum_{n=1}^{\infty} 1/n^2 < \varepsilon_{45}^2 (1 + \int_1^{\infty} (1/n^2) dn) = 2\varepsilon_{45}^2 = A_5 .$$

In the next line we have

$$e^{-A_5} = A_6 ,$$

and the line below yields

$$A_4 = A_6^{-1} = e^{A_5} = e^{2\varepsilon_{45}}$$

On page 271, the second line we have, by using the above result

and theorem 262

$$\begin{aligned} \int_{-\pi}^{\pi} f^4(e^{-1/n} e^{i\theta}) d\theta &< 2\pi(1 - e^{-2/n})(1 + 16e^{2\varepsilon_{44} + 4/5} C_6(\varepsilon_{44})(1 - e^{-2/n})^{-(7/5)})^{\varepsilon_{44}} \\ &< 2(1 + 16e^{2\varepsilon_{44} + (4/5)} C_6(\varepsilon_{44}))n^{(2/5) + \varepsilon_{44}}. \end{aligned}$$

Putting together (322) and (323) we obtain

$$\begin{aligned} \sum_m \int_m |f^{53}(x)| d\theta &< (11 + 3C_{18}^{1/6}(\varepsilon_m))^{49} \cdot 2\pi \cdot (1 + 16e^{2\varepsilon_{44} + (4/5)} C_6(\varepsilon_{44})) \\ &\quad n^{(51/5) + (49\alpha/16) + 49\varepsilon_m + \varepsilon_{44}} \\ &< 105 \cdot 10^{44} [C_6(\varepsilon_{38}) C_6(\varepsilon_{23}) C_6(\varepsilon_{27}) C_{19}(\varepsilon_{12})]^{49/16} \\ &\quad \cdot C_6(\varepsilon_{44}) n^{(51/5) - (49\alpha/16) + 49\varepsilon_m + \varepsilon_{44}}, \end{aligned}$$

where

$$\begin{aligned} 49\varepsilon_m &= (49/16) \cdot [((1/5) + \varepsilon_{27})\varepsilon_{11} + \max(15\varepsilon_{27} + (1-\alpha)\varepsilon_{12}, 16\varepsilon_{27} + \alpha\varepsilon_{12})] \\ &= (392\varepsilon_1/16) + \varepsilon_{49} . \end{aligned}$$

$$\alpha/16 = .0124999$$

$$49((1/5) - (\alpha/16)) + (2/5) = (48/5) - (59/10)\varepsilon_1 - \varepsilon_{49} - C_{31} .$$

$$(49\alpha/16) - (3/5) = (59/10) + \varepsilon_{49} + C_{31} .$$

$$.0124991 = (59/10\varepsilon_1) + \varepsilon_{49} + C_{31} .$$

Let

$$C_{31} = 1/200 .$$

$$\varepsilon_{49} = .001941 .$$

$$\varepsilon_{49} = (49/16) [\varepsilon_{27}(8/500) + \text{Max}(15\varepsilon_{27} + (1-\alpha)\varepsilon_{12}, 16\varepsilon_{27} + \alpha\varepsilon_{12})] .$$

$$\varepsilon_{27} = 1/250,000 .$$

$$\varepsilon_{12} = .001744608/.0124999 \cdot 49 .$$

To evaluate $C_6(1/500)$:

$$C_6(1/500) = \bigwedge_{p < 2^{500}} \text{Max} (l+1/p)^{l/500} .$$

The fraction $(l+1/p)^{l/500}$ decreases as p increases .

$$2^{500} < 10^{150.5150} .$$

$$(l+1/p)^{l/500} < 1.01 ,$$

if

$$p > 10^{148.3544} .$$

$$(l+1/p)^{l/500} < 716/2 \cdot 143 < 10^{2.4245}$$

$$C_6(1/500) < 10^{.44 \cdot 10^{148.6}} + 10^{2.43 \cdot 10^{148.35}}$$

$$< 10^{10^{149}} .$$

$$C_{19}(\varepsilon_{12}) < 2.49(1250/175) < 700 .$$

$$r(n) - (1/2\pi i) \sum_m \int f^{53}(x)/x^{n+1} dx < 735 \cdot 10^{46+3 \cdot 10^{149} \cdot 10^{3/16}} \cdot n^{(48/5) - (1/200)},$$

$$< (1/8) 10^{150} \cdot 9.595 \cdot n .$$

By referring the theorem 278 in this paper

$$\psi_e(x) < C_5 n^{(16/85) + \varepsilon_n} .$$

From the ninth line of page 272 of the book we have

$$(329) \sum_M \int_M f^{53}(x) - \psi_e^{53}(x) d\theta \leq 2^{53} C_5^{53} (\varepsilon_M)^n (348/85) + 53 \varepsilon_M \sum_M \int_M d$$

$$+ 2C_5 n^{(16/85) + \varepsilon_M} \sum_{MM} \int |\psi_e^{52}(x)| d\theta .$$

Relation (330) can be copied exactly from the book. Three lines below, by using theorem 269 in this paper, and by using (329) we

obtain

$$\begin{aligned} |\Psi(x)| &= \Gamma(6/5) |S_e/q| |1-x|^{-1/5} < \Gamma(6/5) q^{-1/5} ((1/n^2) + \epsilon^2)^{-.1} . \\ \int_M |\Psi e^{s-1}(x)| d\theta &< (\Gamma(6/5))^{52} 2^{988} q^{-10.4} \int ((1/n^2)^{-5.2} d\theta \\ &< 4(\Gamma(6/5))^{52} 2^{988} q^{-10.4} n^{9.4} . \\ (331) \quad \sum_M \int_M |\Psi e^{52}(x)| d\theta &< 4(\Gamma(6/5))^{52} 2^{988} n^{9.4} . \end{aligned}$$

Putting (329) and (331) together we obtain

$$\begin{aligned} (332) \quad \sum_M \int_M f^{53}(x) - x^{53}(x) d\theta &< 2^{53} C_5^{53}(\epsilon_M)^4 n^{(763/85)+2\alpha+53\epsilon_M} \\ &\quad + 4(\Gamma(6/5))^{52} 2^{989} C_5 n^{(783/85)+\epsilon_M} \\ (848/85) - 1 + 2\alpha + 53\epsilon_M &= (53/5) - 1 - .005 . \\ 53\epsilon_M &> 3.63/17 , \\ \epsilon_M &> 3.63/901 , \\ \epsilon_M &= .1\epsilon_1 + \epsilon_{46} \end{aligned}$$

where ϵ_{46} is not a function of ϵ_1 .

$$\begin{aligned} \epsilon_1 &= .01 \\ C_6(.01) &< (2^{.01} \cdot 100 / .693)^{10^{31}} \\ &< 10^{10^{32}} \\ [C_6(.01)]^{159/16} &< 10^{10^{33}} \\ \epsilon_{22} &= .01 \end{aligned}$$

and ϵ_{22} is the smallest ϵ for which we have to compute C_{19}

$$\begin{aligned} C_{19}(\epsilon_{22}) &< 200 \\ C_{19}(\epsilon_{22})^{53/16} &< 10^{10^3} \end{aligned}$$

$$\sum_M \int_M |f^{53}(x) - \Psi e^{53}(x)| d\theta < 10^{10^{34}} n^{9.595} .$$

From this inequality and (326) we can get

$$(334) \quad \left| r(n) - \frac{1}{(2\pi i)} \sum_M (\psi_e^{53}(x)/x^{n+1}) dx \right| < (1/4) 10^{10^{150}} n^{9.595}.$$

Second Step: On the top of page 274 of the book is a relation involving B_{16} which we must compute:

$$\begin{aligned} & \left| \int_M (\psi_e^{53}(x)/x^{n+1}) dx - \int_{\mathbb{R}} (\psi_e^{53}(x)/x^{n+1}) dx \right| \\ & < \Gamma(6/5) e^{-53 \cdot 2^{1007} q^{-10.6}} \int_{q^{1-\alpha}}^{q^{1-\alpha} d\theta/\theta} \theta^{10.6} \\ & < (5\pi^{-9.6}/48) (6/5) e^{-53 \cdot 2^{1007} q^{-1} n^{9.6(1-\alpha)}} \end{aligned}$$

from which we derive

$$\begin{aligned} & \left| \sum_M \int_M \psi_e^{53}(x)/x^{n+1} dx - \sum_{M \in \mathbb{R}} \int_M \psi_e^{53}(x)/x^{n+1} dx \right| \\ & < (5\pi^{-9.6}/48) \Gamma(6/5) e^{-53 \cdot 2^{1007} n^{9.3-8.6\alpha}} \\ (335) \quad & \left| r(n) - \frac{1}{(2\pi i)} \sum_{M \in \mathbb{R}} \int_M \psi_e^{53}(x)/x^{n+1} dx \right| < (3/8) 10^{10^{150}} n^{9.595}. \end{aligned}$$

Fourth Step: From the results in this paper, we have

$$\left| ((10.6+n)/n!) - n^{48/5} \right| < 775,200,000 n^{43/5}$$

Using theorem 270 we have

$$\begin{aligned} & (\Gamma(10.6+n) (\Gamma(1.2)^{53}/(n! \Gamma(10.6))) \sum_{q \leq n} \alpha(q) - n^{48/5} (\Gamma(1.2)^{53}) / \Gamma(10.6) \sum_{q \leq n} \alpha(q)) \\ & < 2^{1007} \cdot 740.6 \cdot (\Gamma(1.2))^{53} n^{9.6}. \end{aligned}$$

Fifth Step:

$$\begin{aligned} & \left| \sum_{q > n} \alpha(q) \right| \leq \sum_{q > n} 2^{1007} q^{-9.6} < 2^{1007} q^{-8.6\alpha} \\ (318) \quad & \left| r(n) - ((\Gamma(1.2))^{53} / \Gamma(10.6) n^{9.6}) \right| < .5 \cdot 10^{10^{150}} n^{9.595}. \end{aligned}$$

V. EVALUATION OF b_4

We begin by evaluating b_{20} , page 299,

Theorem 317: For $p \neq m$,

$$A(p) < 4^{53}.$$

The constant b_{20} is the number of terms in

$$\sum \tau_{\psi_1} \dots \tau_{\psi_{53}}$$

as $\psi_1, \dots, \psi_{53}$ range independently over the $(\delta-1)$ special characters $\psi \neq \psi_0$. The symbol δ is defined on the bottom of page 295 in Theorem 309 as $(5, p-1)$, and is therefore not > 5 .

$$\therefore b_{20} \leq 4^{53}.$$

From Theorem 318, page 300,

$$b_{21} = \text{Max} (b_{20}, b_{22}) .$$

From Theorem 311, page 297, for $p = q$

$$|S_e| \leq 4\sqrt{p}.$$

$$A(p) < p(4/\sqrt{p})^{53} = b_{22}p^{51/2}.$$

$$b_{22} = 4^{53} = b_{21}.$$

From Theorem 320, bottom of page 300, for $p/10$, p , a prime ,

$$\sum_{\ell=1}^5 p^{\ell} \leq \sum_{\ell=1}^5 5^{\ell} = c_{43} = 3905 < b_{24}5^{26}$$

$$\leq (3905 \cdot 5^{26} + 1)p^{-26}.$$

$$\begin{aligned} b_{23} &= \max (b_{20}, b_{24}) \\ &= 4^{53} . \end{aligned}$$

From Theorem 321, page 301, it is evident that

$$\begin{aligned} b_{25} &= \max (b_{23}, b_{26}, b_{27}) . \\ b_{26} &= b_{21} + 1 = 4^{53} + 1 . \end{aligned}$$

For $p/10$,

$$\begin{aligned} |\chi_p - 1| &\leq \sum_{k=1}^5 5 = b_{27} p^{-51/2} \leq 3905 \cdot 5^{51/2} p^{-51/2} . \\ b_{25} &= \max (4^{53}, 4^{53} + 1, 3905 \cdot 5^{51/2}) . \\ &= 4^{53+1} . \end{aligned}$$

In Theorem 322, bottom of page 301 ,

$$b_{28} = \text{Max} (b_{25}, b_{29}) .$$

To get b_{29} use the relation $p \leq 5$

$$\begin{aligned} 0 &> 1 - b_{29} p^{-25.5} . \\ b_{29} &= 5^{25.5} + 1 . \\ b_{28} &= 4^{53} + 1 . \end{aligned}$$

We have

$$\chi_p > 1 - (4^{53} + 1)p^{-51/2} .$$

For

$$p > b_{18}, \chi_p > 1 - p^{-2} .$$

To solve for b_{18}

$$\begin{aligned} (4^{53} + 1) b_{18}^{-25.5} &= b_{18}^{-2} . \\ b_{18}^{23.5} &= (4^{53} + 1) . \\ \log (4^{53} + 1) &< 31, 90971 . \\ (2/47) \log (4^{53} + 1) &< 1, 358 < \log 23 . \\ b_{18} &< 23 . \end{aligned}$$

To compute

$$\overline{p \leq b_{18}} \chi_p$$

look back to page 291, bottom of page.

If $\beta = 0$,

$$\chi_p(n) = P^{-52} N(P, n);$$

if $\beta > 0$

$$\chi_p(n) \geq P^{-52} N(P, 0). \quad \textcircled{H}$$

On page 282 we find the definition of p . In our case

$$P(2) = 4 \quad P(5) = 25$$

otherwise

$$P = p.$$

On page 280 we find the definition $N(p^\ell, n)$.

Lemma.

If $(5, p-1) = 1$, and p is a prime, then $N(p, n) \geq p^{52} - 1$.

Proof:

Because n is used only in a congruence, without limiting the generality we may assume

$$0 \leq n \leq p - 1.$$

The number of k th power residues mod p is $(p-1)/(p-1, 5)$; therefore in this case there are $(p-1)$ of them. Consider any number

H_1 such that

$$H_1 = \sum_{v=1}^{52} h^k, \quad 0 \leq h < p.$$

Since the values $1^5 \dots (p-1)^5$ are congruent in some order to the complete set of residues mod p , and $p^5 \equiv 0 \pmod{p}$, there is

one and only one $h_{53} < p$ such that

$$H_1 + h_{53}^5 \equiv n \pmod{p}.$$

There are p^{52} (not necessarily distinct) values for H_1 as each h_r ($r = 1 \dots 52$) ranges independently over the values $1 \dots p$.
 $\therefore$ There are p^{52} congruences such as the one at the top of this page which each has a unique solution h_{53} .

$$\therefore \sum_{r=1}^{53} h^5 \equiv n \pmod{p}$$

for a given n , has p^{52} sets of solutions. If $n \not\equiv 0 \pmod{p}$, not all the h_r are divisible by p and

$$N(p, n) = p^{52}.$$

If

$$n \equiv 0 \pmod{p},$$

there is only one solution where every h_r is divisible by p , i.e., the one where every $h_r = 0$. Then

$$N(p, n) = p^{52} - 1.$$

This lemma takes care of $P=3, 7, 13, 17, 19$, but does not take care of $P = 4, 11, 25$. In the case $P = 4$, there are two fifth power residues, 1, 3. Every residue mod 4, can be represented at least twice as the sum of two fifth power.

For every number

$$H_2 = \sum_{r=1}^{51} h^5,$$

the congruence

$$H_2 + h^5 + h^5 \equiv n \pmod{4}$$

can for every n be satisfied by at least two pairs of values h_{52}, h_{53} , such that not both h_{52} and h_{53} are even.

$$\therefore \sum_{r=1}^{53} h^5 \equiv n \pmod{4}, \quad (0 \leq h < 4)$$

has at least $2 \cdot 4^{51}$ values

$$\therefore N(4, n) \geq 2 \cdot 4^{51}.$$

Consider $p = 11$.

The only fifth power residues are 1 and 10.

If $n \equiv 5$, or $n \equiv 6 \pmod{11}$,

$$\sum_{r=49}^8 h^5 \equiv n \pmod{11}$$

has 3125 solutions since

$$1 + 1 + 1 + 1 + 1 = 5,$$

$$10 + 10 + 10 + 10 + 10 \equiv 6 \pmod{11},$$

and each of $h_r^5 \equiv 1 \pmod{11}$

$$h_r^5 \equiv 10 \pmod{11}$$

has exactly 5 solutions.

Similarly it can be found that 1, 2, 3, 4, 7, 8, 9, 10, each can be represented modulo 11 in more than 3125 ways as the sum of 5 fifth powers. There are 19251 way of representing $0 \pmod{11}$ only one of which requires every $h \equiv 0$.

$$\therefore N(11, n) \geq 3125 \cdot 11^{48}.$$

For $P = 25$, each congruence

$$h_r^5 \equiv 0, 1, 7, 18, 24 \pmod{25}$$

has exactly five solutions, and there are no other numbers which satisfy such a congruence.

Every residue mod 25 can be represented as $\sum_{h=51}^{53} h^5$ in at least 125 ways, such that every h_r is not divisible by 5.

$$\therefore N(25n) \geq 125 \cdot 25^{50}.$$

To evaluate for $p > 19$ the product, $\prod_p^\chi$ we must evaluate for $p \geq 23$ the product $\prod(1-p^{-2})$ which is greater than term by term, for $q \geq 23$ the product $\prod(1-q^{-2})$ where q ranges over all integers.

$$\lim_{x \rightarrow \infty} \prod_{x^2 q \geq 23} (1-q^{-2}) = 22/23.$$

$$b_4 = (22/23)(1/2)(1/5)(3125/11^4) \prod_w ((w^{52}-1)/w^{52}) \geq (625/30613)999 > 1/50,$$

where

$$w = 3, 7, 13, 17, 19.$$

VI. CONCLUSION

Using (318) of this paper we obtain

$$\left| r(n) - \left(\left(\Gamma(6/5) \right)^{53} / \Gamma(53/5) \right) n^{9.6} \right| < 10^{10^{150}} n^{9.595} .$$

$$r(n) > -10^{10^{150}} n^{9.595} + \left(\left(\Gamma(6/5) \right)^{53} / 50 \Gamma(53/5) \right) n^{9.6} .$$

Therefore

$$r(n) \geq 0$$

if

$$(10^{10^{150}} \cdot 50 \Gamma(53/5) / (6/5)^{53})^{200} \leq n$$

$$10^{2 \cdot 10^{152} + 10^{11}} \leq n$$

$$10^{10^{153}} \leq n .$$

VITA

Ruth Glidden Mason was born in Chicago on April 24, 1906. She attended Knickerbocker Grammar School and Robert Waller High School, which are public schools in Chicago. In 1922-23 she attended the University of Chicago. For the next three years she was a student at Wellesley College where she received a B. A. degree in 1926. Her principal instructors in Mathematics there were Professors Merrill and Young. In the years 1927-32, Ruth Mason spent many quarters studying Mathematics at the University of Chicago. That institution granted her a Master of Science degree in 1928. Besides studying at the University of Chicago, she spent one semester in 1930 at the University of California where she took a course under Professor Lehmer.

She wishes to acknowledge her indebtedness to all of her instructors, Professors Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon, Lunn, Mac Millan, Slaught, Bell, Bompiani, and Mac Duffee, and especially to Professor Dickson under whose direction this thesis has been written.

